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THEOREM ON $2n$ INTERVALS IN A TIME-OPTIMAL PROBLEM WITH MAGNITUDE- AND IMPULSE-CONSTRAINED CONTROL

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We examine controlled systems which are described by linear differential equations with constant coefficients. We assume that the controlling forces are constrained simultaneously in magnitude and in impulse. The time-optimal problem for this case was investigated, for example, in [1 – 3].

Below we prove a theorem on $2n$ intervals of constancy of the optimal control. This theorem is analogous to the theorem on n intervals given in [4, 5], which holds when the control is bounded only in magnitude.

1. Statement of the problem. We consider a controlled system described by a linear matrix differential equation with real constant coefficients

$$dx/dt = Ax + bu \tag{1.1}$$

Here $x = \| x_i \|$, $A = \| a_{ij} \|$, $b = \| b_i \|$ are matrices of order $(n \times 1)$, $(n \times n)$, $(n \times 1)$, respectively, $u = u(t)$ is a scalar piecewise-continuous time function satisfying simultaneously the two constraints

$$|u(t)| \leqslant M \qquad (M = \text{const} > 0) \tag{1.2}$$

$$\int_0^\infty |u(\tau)|\, d\tau \leqslant N \qquad (N = \text{const} > 0) \tag{1.3}$$

Constraints (1.2) and (1.3) are simultaneously present, for example, when control is effected by a jet thruster. Here inequality (1.2) corresponds to the boundedness of the fuel flow rate, while inequality (1.3) corresponds to the boundedness of the thruster's propellant capacity. We denote by Ω the set of piecewise-continuous functions $u(t)$ satisfying simultaneously inequalities (1.2) and (1.3). We examine the time-optimal problem of taking system (1.1) to the origin by means of a control $u(t) \in \Omega$.

2. The optimal control. The solution of Eq. (1.1) has the form

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}\,bu(\tau)\,d\tau \tag{2.1}$$

where $x(0)$ is the system's initial state. Let $x(t) = 0$ at $t = T$, then from (2.1) we have

$$-x(0) = \int_0^T e^{-A\tau} bu(\tau)\,d\tau \tag{2.2}$$

If equality (2.2) is realized by admissible control $u(t) \in \Omega$, then this control satisfies the inequality

$$\int_0^T |u(\tau)|\,d\tau \leqslant N \tag{2.3}$$

By $\Omega^*(T)$ we denote the set of piecewise-continuous functions $u(t)$ satisfying conditions (1.2) and (2.3). We introduce the notation

$$v(T) = \int_0^T e^{-A\tau} bu(\tau)\,d\tau$$

and in the phase space X_n we consider the attainability region

$$Q(T) = \{v(T):\ \ u(t) \in \Omega(T)\}$$

This region possesses a number of properties, described in [3, 6]. We take an arbitrary unit vector η $(1 \times n)$ and we construct support hyperplanes of set $Q(T)$, orthogonal to vector η. There are two such planes and they are symmetric to each other relative to the origin. We denote by $\Pi(\eta,\ T)$ that one of the planes for which the vector η at the point of tangency is the outward normal to set $Q(T)$ (Fig. 1).

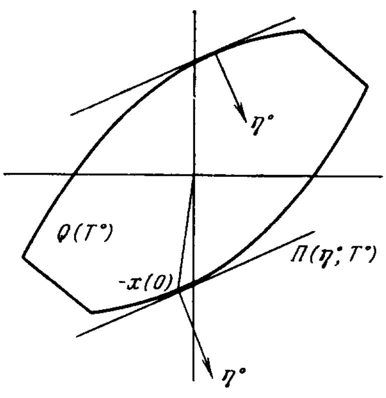

Fig. 1

Let T° be the minimum time in which system (1.1) can be taken from state $x(0)$ to the origin, i.e. the time-optimal time. If point $x(0)$ belongs to the system's controllability region [6], then the minimum time T° exists for it. The point $x(0)$ (as well as $-x(0)$) belongs to the boundary of the region $Q(T^\circ)$ [3, 5]. We construct the support hyperplane $\Pi(\eta^\circ,\ T^\circ)$ of set $Q(T^\circ)$, containing the point $-x(0)$ (Fig. 1). If the optimal transient time T° corresponding to state $x(0)$ satisfies the inequality

$$MT^\circ \leqslant N \tag{2.4}$$

then the optimal control has the form [5]

$$u^\circ(t) = M\ \mathrm{sgn}\ [\eta^\circ e^{-At}\,b] \tag{2.5}$$

Under condition (2.4) constraint (1.3) is unessential.

Assume that the inequality

$$MT^\circ > N \tag{2.6}$$

holds. In this case the optimal control is unique and has the form [3]

$$u^\circ(t) = \begin{cases} M \operatorname{sgn}[\eta^\circ e^{-At} b] & \text{for } \ t \in E(\sigma^\circ) \\ 0 & \text{for } \ t \in G(\sigma^\circ) \end{cases} \tag{2.7}$$

Here

$$E(\sigma) = \{t \in [0, \ T^\circ]: |\eta^\circ e^{-At} b| > \sigma\}$$
$$G(\sigma) = \{t \in [0, \ T^\circ]: |\eta^\circ e^{-At} b| \leqslant \sigma\} \ \ (E(\sigma) \cup G(\sigma) = [0, \ T])$$

The quantity σ° satisfies the equation

$$\mu E(\sigma) = N/M \tag{2.8}$$

in which $\mu E(\sigma)$ is the Lebesgue's measure [7] of set $E(\sigma)$.

If the support hyperplane $\Pi(\eta^\circ, \ T^\circ)$ is unique, then the inequality

$$\varphi(t) = \eta^\circ e^{-At} b \not\equiv \text{const} \tag{2.9}$$

holds [3] under condition (2.6). Let us assume that the support plane at the point $-x(0)$ is not unique ($-x(0)$ is a corner point of region $Q(T^\circ)$). In this case, among all the support planes containing point $-x(0)$ we can select a plane $\Pi(\eta^\circ, T^\circ)$ such that the inequality (2.9) also holds for the corresponding vector η°. In what follows we consider that the vector η° has been chosen in that manner.

From the analyticity of the function $\varphi(t)$ it follows that under condition (2.9) the number l of local maxima of the function $|\varphi(t)|$ on the interval $[0, \ T^\circ]$ is finite. Here the set $E(\sigma^\circ)$ consists of a finite number of intervals (in Fig. 2 this set is shown by the heavy lines) the sum of whose lengths equals N/M in correspondence with Eq. (2.8). The set $G(\sigma^\circ)$ consists of a finite number of segments. From expression (2.7) it follows that the optimal control $u^\circ(t)$ has a finite number of intervals on which $|u^\circ(t)| \equiv M$ and a finite number of intervals on which $u^\circ(t) \equiv 0$; in other words, the control $u^\circ(t)$ has a finite number of intervals of constancy.

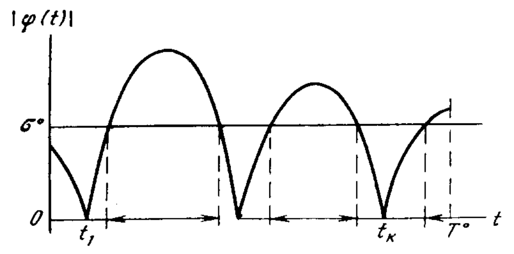

Fig. 2

3. Theorem on $2n$ intervals. The assertion on the finiteness of the number of intervals of constancy of the optimal control holds for systems with any eigenvalues. A stronger assertion holds for systems with real eigenvalues.

Theorem. If all the eigenvalues of matrix A are real, then for any initial conditions the optimal control $u^\circ(t)$ has no more than n intervals on which $|u^\circ(t)| \equiv M$ and no more than n intervals on which $u^\circ(t) \equiv 0$. Consequently, the control $u^\circ(t)$ has no more than $2n$ intervals of constancy.

Let us prove this theorem. If the initial state $x(0)$ is such that the minimum time T° for it satisfies inequality (2.4), then the optimal control $u^\circ(t)$ of (2.5) does not have intervals of complete measure, on which $u^\circ(t) \equiv 0$. In accordance with Fel'dbaum's theorem [4, 5] the number of intervals on which $|u^\circ(t)| \equiv M$ does not exceed the number n.

Now let the initial condition $x(0)$ be such that the minimum time T° for it satisfies inequality (2.6). The number of intervals of set $E\ (\sigma^\circ)$ does not exceed the number l of local maxima of function $|\ \varphi\ (t)|$ on the interval $[0,\ T^\circ]$ (Fig. 2). Let $0 \leqslant t_1 < t_2 < \ldots < t_k \leqslant T^\circ$ be k zeros of the function $\varphi\ (t)$. We denote by m the number of points on the interval $[0,\ T^\circ]$ at which the derivative $\varphi'\ (t)$ equals zero. If all the eigenvalues of matrix A are real, then $k \leqslant n-1$ and $m \leqslant n-1$ [5]. The local maxima of function $|\ \varphi\ (t)\ |$ can occur at points where $\varphi'\ (t) = 0$, as well as at the points $t = 0$ and $t = T^\circ$ (Fig. 2). Consequently, $l \leqslant m + 2 \leqslant n + 1$. We now prove that $l \leqslant n$.

Assume that $l = n + 1$. Then the function $|\ \varphi\ (t)\ |$ has local maxima at $m = n - 1$ points of the interval $(0,\ T^\circ)$ and at the two points $t = 0$ and $t = T^\circ$; here $t_1 > 0$ and $t_k < T^\circ$. In this case the function $|\ \varphi\ (t)\ |$ has no local maxima on the interval $[0,\ T^\circ]$ excepting the points $t_1, \ldots,\ t_k$. There are no local maxima on the intervals $(0,\ t_1)$, $(t_k,\ T^\circ)$ because otherwise there would be local minima on these intervals. For this same reason there can be only one maximum on each of the $k-1$ intervals $(t_1,\ t_2), \ldots,\ (t_{k-1},\ t_k)$ (Fig. 2). Then we obtain that $m = k - 1 \leqslant n-2$, but this contradicts the equality $m = n - 1$. Thus, $l \leqslant n$; the first part of the theorem is proved.

If $l = n$, then the function $|\ \varphi\ (t)\ |$ reaches a local maximum either for $t = 0$ or for $t = T^\circ$ Here the set $G\ (\sigma^\circ)$ on which $u^\circ\ (t) \equiv 0$ consists of no more than n segments. This assertion is all the more valid when $l < n$. The theorem is proved.

If we examine, for example, the first-order equation $x_1^{\boldsymbol\cdot} = -\ x_1 + u$, then for points $x_1(0)$ sufficiently distant from zero the optimal control in this equation has exactly two intervals of constancy. As regards, for example, the second-order equation $x^{\boldsymbol{\cdot\cdot}} = u$, considered in [3], the number of intervals of constancy for it under all initial conditions does not exceed three $(3 = 2n - 1)$.

The situation is different when only the constraint (1.2) is present. For any completely controllable [8] system with real eigenvalues there exist initial states for which the optimal control has exactly n intervals of constancy [4]. We note that in systems with a control constrained only in impulse (inequality (1.3)) a theorem, "contiguous" to the one proved here, on the number of impulses holds [9 – 11].

If system (1.1) is not completely controllable [8], i.e.

$$\text{rank}\ \|\ b,\ Ab, \ldots,\ A^{n-1}b\ \| = \rho < n$$

then by a nonsingular transformation of the form $y = Kx$ it can be brought to the following form [12]:

$$dy_1/dt = A_{11}y_1 + A_{12}y_2 + b_1u, \qquad dy_2/dt = A_{22}y_2 \qquad (3.1)$$

Here y_1 and y_2 are column-vectors of order $(\rho \times 1)$ and $((n-\rho) \times 1)$, respectively. The matrices A_{11}, A_{12}, A_{22}, b_1 have appropriate orders. Here

$$\text{rank}\ \|b_1,\ A_{11}b_1, \ldots,\ A_{11}^{\rho-1}b_1\| = \rho$$

System (3.1) can be led to the origin only from those points $y\ (0)$ at which $y_2\ (0) = 0$. Using the theorem proved above, we can assert that if all the eigenvalues of matrix A_{11} are real, then the optimal control in system (3.1) has no more than 2ρ intervals of constancy for all initial conditions.

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HYDRODYNAMIC INTERACTION BETWEEN BODIES IN A PERFECT INCOMPRESSIBLE FLUID AND THEIR MOTION IN NONUNIFORM STREAMS

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A general method based on the use of Lagrangian equations for determining hydrodynamic interaction between bodies in a fluid is presented. Formulas for the kinetic energy and the Lagrangian function are reduced to a form which permits an effective application of the method of small parameter. Additive components of kinetic energy and of the Lagrangian function, which determine the hydrodynamic interaction between two bodies, one of which is small in comparison with the distance between the two, are calculated. The method is used for considering the case of several bodies. The results are expressed in terms of coefficients of apparent mass of individual bodies in a boundless fluid. General